

Gorenstein homological complexes with respect to duality pairs*

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Abstract: Let $(\mathcal{X}, \mathcal{Y})$ be a duality pair of complexes. In this article, we first introduce and investigate the notion of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes. Moreover, we establish a relationship between a Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex and its terms for some special duality pairs $(\mathcal{X}, \mathcal{Y})$. Finally, we obtain some equivalent characterizations of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective dimensions of complexes.

Key words: Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex; Homological property; Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective dimension

CLC number: O153.3 **Document code:** A **Article ID:** 2097 - 0137(XXXX)XX - 0001 - 19

In the 1990s, Enochs et al. (1995) introduce the concept of Gorenstein injective modules. A module M is said to be Gorenstein injective if there is an exact sequence

$$\dots \rightarrow I^{-1} \xrightarrow{\delta^{-1}} I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \dots$$

of injective modules such that $M = \text{Ker } \delta^0$ and for any injective module I , $\text{Hom}_R(I, -)$ leaves the exact sequence above exact. Dually, the concept of Gorenstein projective modules is also introduced. Later, Enochs et al. (1998) extend the concepts above to the category of complexes, introduce the notions of both Gorenstein injective complexes and Gorenstein projective complexes.

As we know, in the category of complexes, the relationship between complexes and their terms is an important research topic, and has attracted the attention of many scholars. For example, a complex C is DG-injective if and only if C is a complex consisting of injective modules and $\text{Hom}_R(E, C)$ is exact for every exact complex E according to García Rozas(1999). A complex X is Gorenstein injective (resp., Gorenstein projective) if and only if each X_n is a Gorenstein injective (resp., Gorenstein projective) module as shown in Yang et al. (2011). A complex X is Ding injective if and only if each term X_n is a Ding injective module over coherent rings according to Yang et al. (2020). Recently, this was further developed in Gillespie et al. (2023), where it was proved that the same results hold over any ring.

The notion of duality pairs is introduced by Holm et al. (2009). Since the duality pair is closely related to the existence of purity, covers and envelopes, and cotorsion pairs, it plays an important role in homological algebra. There are many duality pairs in the category of modules; see, for example, Bravo et al. (2018); Gillespie (2019). In particular, the flat modules and injective modules form a duality pair. Furthermore, Yang(2011)

* Received: 2025 - 01 - 25

Accepted: 2026 - 01 - 23

Published online: XXXX - XX - XX

Supported by Young Talents Team Project of Gansu Province(2025QNTD49);

National Natural Science Foundation of China(12061061);

Fundamental Research Funds for the Central Universities(31920260074);

Lanshan Talent Project of Northwest Minzu University(Xbmulsr202412);

University-level Innovation Team Project of Northwest Minzu University(10017633, 11087633)

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ZR20250021

extends the notion to the category of complexes, introduces the concept of a duality pair of complexes.

Recently, Gillespie(2019) introduce and study the concept of Gorenstein $(\mathcal{L}, \mathcal{A})$ -injective modules, where $(\mathcal{L}, \mathcal{A})$ is a complete duality pair of modules. A module M is said to be Gorenstein $(\mathcal{L}, \mathcal{A})$ -injective if there is an exact sequence

$$\dots \rightarrow I^{-1} \xrightarrow{\delta^{-1}} I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \dots$$

of injective modules such that $M = \text{Ker } \delta^0$ and $\text{Hom}_R(A, I)$ remains exact for all $A \in \mathcal{A}$. It is well known that extending the homological theory from the category of modules to the category of complexes is a very natural and important subject in homological algebra. In this paper, we introduce and study the notion of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes, where $(\mathcal{X}, \mathcal{Y})$ is a duality pair of complexes.

As we know, a complex C is DG-injective if and only if C_n is an injective module for each $n \in \mathbb{Z}$ and $\text{Hom}_R(E, C)$ is exact for every exact complex E .

1 Preliminaries

Throughout this article, R denotes an associative ring with unity. A complex C of left R -modules will be denoted

$$\dots \rightarrow C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \dots$$

Unless stated otherwise, by the term "module" we always mean a left R -module and the term "complex" we always mean a complex of left R -modules.

We will use superscripts to distinguish complexes. So if $\{C^i\}_{i \in \mathbb{Z}}$ is a family of complexes, C^i will be

$$\dots \rightarrow C_1^i \xrightarrow{\delta_1^i} C_0^i \xrightarrow{\delta_0^i} C_{-1}^i \xrightarrow{\delta_{-1}^i} \dots$$

Given a complex C and an integer n , the n th suspension of C , denoted $\Sigma^n C$, is the complex given by $(\Sigma^n C)_k = C_{k-n}$ and $\delta_k^{\Sigma^n C} = (-1)^n \delta_{k-n}$. The n th cycle module is defined as $\text{Ker } \delta_n$ and is denoted by $Z_n(C)$. The n th boundary module is defined as $\text{Im } \delta_{n+1}$ and is denoted by $B_n(C)$. For a module M , we denote the n -disk on M by $D^n(M)$. This is the complex consisting only of $M \xrightarrow{1_M} M$ concentrated in degrees n and $n-1$. We denote the n -sphere on M by $S^n(M)$, and this is the complex consisting of M in degree n and 0 elsewhere.

The category of complexes of left R -modules (resp. , right R -modules) is denoted by $\mathcal{C}(R)$ (resp. , $\mathcal{C}(R^r)$). This category has enough projectives and injectives. This can be seen from the fact that any complex of the form

$$\dots \rightarrow 0 \rightarrow M \xrightarrow{\text{id}} M \rightarrow \dots$$

with M projective (resp. , injective) is projective (resp. , injective).

Let C and D be complexes. We use $\text{Hom}(C, D)$ to denote the abelian group of morphisms from C to D and Ext^i for $i \geq 0$ will denote the groups we get from the right derived functor of Hom . Recall that $\text{Ext}^1(C, D)$ is the group of equivalence classes of short exact sequences $0 \rightarrow D \rightarrow Z \rightarrow C \rightarrow 0$. We let $\text{Ext}_{\text{dec}}^1(C, D)$ be the subgroup of $\text{Ext}^1(C, D)$ consisting of those short exact sequences which are split in each degree. We denote by $\text{Hom}_R(C, D)$ the complex of abelian groups with $\text{Hom}_R(C, D)_n = \prod_{i \in \mathbb{Z}} \text{Hom}_R(C_i, D_{n+i})$ and such that if $f \in \text{Hom}_R(C, D)_n$ then $(\delta_n(f))_m = \delta_{m+n}^D f_m - (-1)^n f_{m+1} \delta_m^C$.

Given two complexes C and D . Let $\underline{\text{Hom}}(C, D) = Z(\text{Hom}_R(C, D))$. We then see that $\underline{\text{Hom}}(C, D)$ can be made into a complex with $\underline{\text{Hom}}(C, D)^m$ the abelian group of morphisms from C to $\Sigma^m D$ and with boundary operator given by $f \in \underline{\text{Hom}}(C, D)^m$; then $\delta_m(f): C \rightarrow \Sigma^{m+1} D$ with $\delta_m(f)_n = (-1)^m \delta_{n+m}^D f_n$ for any $n \in \mathbb{Z}$. We note that the new functor $\underline{\text{Hom}}(C, D)$ will have right derived functors whose values will be complexes. These values should certainly be denoted $\underline{\text{Ext}}^i(C, D)$. It is not hard to see that $\underline{\text{Ext}}^i(C, D)$ is the complex

$$\dots \rightarrow \text{Ext}^i(C, \Sigma^{n-1}D) \rightarrow \text{Ext}^i(C, \Sigma^n D) \rightarrow \text{Ext}^i(C, \Sigma^{n+1}D) \rightarrow \dots$$

with boundary operator induced by the boundary operator of D . For a complex C , denote C^+ as $\underline{\text{Hom}}(C, \overline{\mathbb{Q}/\mathbb{Z}})$.

The tensor product of a complex of right R -modules C and a complex of left R -modules D is the complex of \mathbb{Z} -modules $C \otimes D$ with $(C \otimes D)^n = \bigoplus_{i \in \mathbb{Z}} C_i \otimes {}_R D_{n-i}$ and

$$\delta(c \otimes d) = \delta_i^c(c) \otimes d + (-1)^i c \otimes \delta_{n-i}^d(y)$$

for $c \in C_i, d \in D_{n-i}$.

We define $C \overline{\otimes} D$ to be $\frac{(C \otimes D)}{B(C \otimes D)}$. Then with the maps

$$\frac{(C \otimes D)^n}{B^n(C \otimes D)} \rightarrow \frac{(C \otimes D)^{n+1}}{B^{n+1}(C \otimes D)}, \quad x \otimes y \mapsto \delta^c(x) \otimes y,$$

where $x \otimes y$ is used to denote the coset in $\frac{(C \otimes D)^n}{B^n(C \otimes D)}$, we get a complex.

For C a complex of left R -modules we have two functors $-\overline{\otimes} C : \mathcal{C}_R \rightarrow \mathcal{C}_{\mathbb{Z}}$ and $\underline{\text{Hom}}(C, -) : {}_R\mathcal{C} \rightarrow \mathcal{C}_{\mathbb{Z}}$, where \mathcal{C}_R (resp. ${}_R\mathcal{C}$) denotes the category of complexes of right R -modules (resp. left R -modules).

A morphism $f : C \rightarrow D$ of complexes is said to be null-homotopic if there exists a sequence $h_n : C_n \rightarrow D_{n+1}$ such that $f_n = \delta_{n+1}^D h_n + h_{n-1} \delta_n^C$ for all $n \in \mathbb{Z}$. The mappingcone of $f : C \rightarrow D$ is denoted as $M(f) = D \oplus \Sigma C$, where $M(f)_n = D_n \oplus C_{n-1}$ and $\delta_n^{M(f)}(d, c) = (\delta_n^D(d) + f_{n-1}(c), -\delta_{n-1}^C(c))$. Let \mathfrak{L} be a class of objects in an abelian category \mathcal{A} and $L \in \mathfrak{L}$. A morphism $f : D \rightarrow L$ is called an L-preenvelope of D if $\text{Hom}(L, L') \rightarrow \text{Hom}(D, L') \rightarrow 0$ is exact for any $L' \in \mathfrak{L}$.

Definition 1(Yang et al. ,2011) Let \mathfrak{L} be a class of objects in an abelian category \mathcal{A} . We call \mathfrak{X} projectively resolving if all projective objects are contained in \mathfrak{X} and for every short exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ with $X'' \in \mathfrak{X}$, the conditions $X' \in \mathfrak{X}$ and $X \in \mathfrak{X}$ are equivalent. We call \mathfrak{X} injectively resolving if all injective objects are contained in \mathfrak{X} and for every short exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ with $X' \in \mathfrak{X}$, the conditions $X'' \in \mathfrak{X}$ and $X \in \mathfrak{X}$ are equivalent.

Definition 2(Holm et al. ,2009) A duality pair over R is a pair $(\mathcal{A}, \mathcal{B})$, where \mathcal{A} is a class of left R -modules and \mathcal{B} is a class of right R -modules, subject to the following conditions:

- (1) For a left R -module A , one has $A \in \mathcal{A}$ if and only if $\text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \in \mathcal{B}$.
- (2) \mathcal{B} is closed under direct summands and finite direct sums.

Definition 3(Yang,2011) A duality pair over R is a pair $(\mathcal{X}, \mathcal{Y})$, where \mathcal{X} is a class of complexes of left R -modules and \mathcal{Y} is a class of complexes of right R -modules, which are closed under isomorphisms, subject to the following conditions:

- (1) For any complex X , one has $X \in \mathcal{X}$ if and only if $X^+ \in \mathcal{Y}$.
- (2) \mathcal{Y} is closed under direct summands and finite direct sums.

In the following, $(\mathcal{A}, \mathcal{B})$ will always denote a duality pair of modules, $(\mathcal{X}, \mathcal{Y})$ will always denote a duality pair of complexes.

2 Gorenstein injective complexes with respect to duality pairs

In this section, we will investigate the concept of Gorenstein injective complexes with respect to duality pairs and give some equivalence characterizations of Gorenstein injective complexes with respect to duality pairs.

Definition 4 A complex M is said to be Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective if there exists an exact sequence of injective complexes

$$\mathbb{I} : \dots \rightarrow I^{-1} \xrightarrow{\delta^{-1}} I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} I^2 \rightarrow \dots$$

such that

- (i) $M = \text{Ker } \delta^0$;
- (ii) $\text{Hom}(Y, \mathbb{I})$ remains exact for all $Y \in \mathcal{Y}$.

A complex M is said to be Gorenstein $(\mathcal{X}, \mathcal{Y})$ -projective if there exists an exact sequence of projective complexes

$$\mathbb{P} : \dots \rightarrow P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\delta^0} P^1 \xrightarrow{\delta^1} P^2 \rightarrow \dots$$

such that

- (iii) $M = \text{Ker } \delta^0$;
- (iv) $\text{Hom}(\mathbb{P}, X)$ remains exact for all $X \in \mathcal{X}$.

The class of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective and the class of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -projective complexes are denoted by $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$ and $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}(R)$, respectively.

Every conclusion about Gorenstein $(\mathcal{X}, \mathcal{Y})$ -projective complexes is dual to a corresponding conclusion about Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes. Therefore, in the following, we will only discuss the case with respect to Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes.

Remark 1 (i) It is clear that every injective complex is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective.

(ii) If

$$\mathbb{I} : \dots \rightarrow I^{-1} \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$$

is a $\text{Hom}(Y, -)$ exact exact sequence of complexes for all $Y \in \mathcal{Y}$, then by symmetric, all the kernels, the images, and the cokernels of \mathbb{I} are Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes.

Example 1 (i) A complex F is said to be of type FP_∞ if there exists an exact sequence of complexes

$$\dots \rightarrow P^n \rightarrow P^{n-1} \rightarrow \dots \rightarrow P^1 \rightarrow P^0 \rightarrow F \rightarrow 0,$$

where each P^i is finitely generated projective for every $0 \leq i \leq +\infty$. We call a complex A absolutely clean if $\text{Ext}^1(F, A) = 0$ for all complexes F of type FP_∞ . On the other hand, we call a complex L level if $\text{Tor}_1(F, L) = 0$ for all complexes F of type FP_∞ of right R -modules in Bravo et al. (2017).

Let \mathcal{L} denote the class of all level complexes, and \mathcal{AC} denote the class of all absolutely clean complexes, then $(\mathcal{L}, \mathcal{AC})$ is a duality pair (Gillespie, 2017). Thus, Gorenstein $(\mathcal{L}, \mathcal{AC})$ -injective complexes are coincide with Gorenstein AC-injective complexes in Yang et al. (2020), Gorenstein $(\mathcal{L}, \mathcal{AC})$ -projective complexes are coincide with Gorenstein AC-projective complexes in Gillespie(2018).

(ii) The notations \mathcal{I} and \mathcal{F} are used to denote the class of injective complexes and the class of flat complexes, respectively. We notice that $(\mathcal{F}, \mathcal{I})$ is a duality pair. Thus, Gorenstein $(\mathcal{F}, \mathcal{I})$ -injective complexes are coincide with Gorenstein injective complexes in Enochs et al. (1998). Gorenstein $(\mathcal{F}, \mathcal{I})$ -projective complexes are coincide with Ding projective complexes in Gillespie (2017). If R is a perfect ring, Gorenstein $(\mathcal{F}, \mathcal{I})$ -projective complexes are coincide with Gorenstein projective complexes in Enochs et al. (1998).

(iii) Let n be a nonnegative integer, a complex F is said to be of type FP_n if there exists an exact sequence of finitely generated projective complexes

$$P^n \rightarrow P^{n-1} \rightarrow \dots \rightarrow P^1 \rightarrow P^0 \rightarrow F \rightarrow 0.$$

A complex I is said to be FP_n -injective if $\text{Ext}^1(L, I) = 0$ for all complexes L of type FP_n . A complex F of right R -modules is said to be FP_n -flat if $\text{Tor}_1(F, L) = 0$ for all complexes L of type FP_n (Zhao et al. ,2019).

Let \mathcal{FPF}_n denote the class of all FP_n -flat complexes, and \mathcal{FPI}_n denote the class of FP_n -injective complexes. If R is a coherent ring, then $(\mathcal{FPF}_1, \mathcal{FPI}_1)$ is a duality pair (Bravo et al. ,2017; Zhao et al. , 2019). Thus, Gorenstein $(\mathcal{FPF}_1, \mathcal{FPI}_1)$ -injective complexes are coincide with Ding injective complexes in Gillespie(2017).

If $n \geq 2$, $(\mathcal{FPF}_n, \mathcal{FPI}_n)$ is a duality pair. The Gorenstein $(\mathcal{FPF}_n, \mathcal{FPI}_n)$ -injective complexes is particularly called Gorenstein FP_n -injective complexes. The Gorenstein $(\mathcal{FPF}_n, \mathcal{FPI}_n)$ -projective complexes is particularly

called Gorenstein FP_n -projective complexes.

(iv) A complex I is said to be DG -injective if I is a complex consisting of injective modules and if $\text{Hom}_R(E, I)$ is exact whenever E is exact. A complex F is said to be DG -flat if F is a complex consisting of flat modules and if for every exact complex E of right R -modules, $E \otimes F$ is exact (García Rozas, 1999).

Let \mathcal{DF} denote the class of all DG -flat complexes, and \mathcal{DI} denote the class of DG -injective complexes. Then $(\mathcal{DF}, \mathcal{DI})$ is a duality pair (Yang, 2011). The Gorenstein $(\mathcal{DF}, \mathcal{DI})$ -injective complexes is particularly called Gorenstein DG -injective complexes. The Gorenstein $(\mathcal{DF}, \mathcal{DI})$ -projective complexes is particularly called Gorenstein DG -projective complexes

(v) Consider for each integer $n \geq 0$ the following complex class:

$$\mathcal{F}_n = \{F \in \mathcal{C}(R) \mid \text{fd}F \leq n\}, \quad \mathcal{I}_n = \{I \in \mathcal{C}(R^e) \mid \text{id}I \leq n\},$$

where fd denotes the flat dimension, and id denotes the injective dimension. $(\mathcal{F}_n, \mathcal{I}_n)$ is a duality pair (Yang, 2011).

The Gorenstein $(\mathcal{F}_n, \mathcal{I}_n)$ -injective complexes is particularly called Gorenstein FI_n -injective complexes. The Gorenstein $(\mathcal{F}_n, \mathcal{I}_n)$ -projective complexes is particularly called Gorenstein FI_n -projective complexes.

Recall that an exact sequence $\mathbb{X}: \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ in $\mathcal{C}(R)$ with each $X_i \in \mathcal{X}$ is said to be an \mathcal{X} resolution of M . An \mathcal{X} coresolution of M is defined dually. An exact sequence in $\mathcal{C}(R)$ is called $\text{Hom}(\mathcal{Y}, -)$ exact if it remains still exact after applying the functor $\text{Hom}(Y, -)$ for all $Y \in \mathcal{Y}$.

Proposition 1 Let M be a complex. The following are equivalent:

(i) M is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective.

(ii) $\text{Ext}^{\geq 1}(Y, M) = 0$ for all $Y \in \mathcal{Y}$ and M admits a $\text{Hom}(\mathcal{Y}, -)$ exact injective resolution.

(iii) There exists an exact sequence $0 \rightarrow K \rightarrow I \rightarrow M \rightarrow 0$ where $I \in \mathcal{I}$ and $K \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$.

Proof (i) \Rightarrow (ii) Let M be a Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex. By definition 4, there exists an exact sequence of complexes

$$\cdots \rightarrow I^{-1} \rightarrow I^0 \rightarrow M \rightarrow 0,$$

where I^i is an injective complex for each $i \leq 0$, and which remains exact after applying $\text{Hom}(Y, -)$ for any $Y \in \mathcal{Y}$.

Obviously, M admits an injective coresolution $0 \rightarrow M \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots$. Let $K^i = \text{Ker}(I^i \rightarrow I^{i+1})$, $i \geq 1$. By the definition of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes, K^i is a Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex. Applying $\text{Hom}(Y, -)$ for any $Y \in \mathcal{Y}$ to the exact sequence of complexes

$$0 \rightarrow M \rightarrow I^1 \rightarrow K^2 \rightarrow 0,$$

we have $\text{Ext}^1(Y, M) = 0$ and $\text{Ext}^{i+1}(Y, M) \cong \text{Ext}^i(Y, K^2)$, $i \geq 1$. In a similar way, applying $\text{Hom}(Y, -)$ for any $Y \in \mathcal{Y}$ to the exact sequence of complexes $0 \rightarrow K^i \rightarrow I^i \rightarrow K^{i+1} \rightarrow 0$, we get $\text{Ext}^{i+1}(Y, K^i) \cong \text{Ext}^i(Y, K^{i+1})$, $i \geq 1$. Hence $\text{Ext}^i(Y, M) = 0$, $i \geq 1$.

(ii) \Rightarrow (i) It is evident that M admits an injective coresolution $0 \rightarrow M \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots$. Let $K^i = \text{Ker}(I^i \rightarrow I^{i+1})$, $i \geq 1$. Since $\text{Ext}^{\geq 1}(Y, M) = 0$, $0 \rightarrow M \rightarrow I^1 \rightarrow K^2 \rightarrow 0$ remains exact after applying $\text{Hom}(Y, -)$ for $Y \in \mathcal{Y}$ and $\text{Ext}^{\geq 1}(Y, K^2) = 0$. Since $\text{Ext}^{\geq 1}(Y, K^2) = 0$, $0 \rightarrow K^2 \rightarrow I^2 \rightarrow K^3 \rightarrow 0$ also remains exact after applying $\text{Hom}(Y, -)$ for $Y \in \mathcal{Y}$ and $\text{Ext}^{\geq 1}(Y, K^3) = 0$. By repeating the preceding process, we get M is a Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex.

(i) \Rightarrow (iii) This follows directly from the definition of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes.

(iii) \Rightarrow (ii) The equivalence of (i) and (ii) implies that K is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective, $\text{Ext}^{\geq 1}(Y, K) = 0$, and K admits a $\text{Hom}(\mathcal{Y}, -)$ exact injective resolution

$$\cdots \rightarrow I^{-1} \rightarrow I^0 \rightarrow K \rightarrow 0. \quad (1)$$

Note that there exists a long exact sequence

$$\cdots \rightarrow \text{Ext}^i(Y, I) \rightarrow \text{Ext}^i(Y, M) \rightarrow \text{Ext}^{i+1}(Y, K) \rightarrow \cdots,$$

we get $\text{Ext}^{\geq 1}(Y, M) = 0$. Assembling (1) with $0 \rightarrow K \rightarrow I \rightarrow M \rightarrow 0$, we get a $\text{Hom}(\mathcal{Y}, -)$ exact injective resolution $\cdots \rightarrow I^{-1} \rightarrow I^0 \rightarrow I \rightarrow M \rightarrow 0$, as desired. \square

Definition 5 Let $(\mathcal{A}, \mathcal{B})$ be a duality pair of modules such that \mathcal{B} is closed under extensions, M a complex.

(i) M is said to be an \mathcal{A} complex if it is exact and $Z_n(M) \in \mathcal{A}$ for all n .

(ii) M is said to be a \mathcal{B} complex if it is exact and $Z_n(M) \in \mathcal{B}$ for all n .

We denote the class of \mathcal{A} complexes by $\tilde{\mathcal{A}}$. Similarly, the \mathcal{B} complexes are denoted by $\tilde{\mathcal{B}}$.

Lemma 1 Let $(\mathcal{A}, \mathcal{B})$ be a duality pair of modules such that \mathcal{B} is closed under extensions. Then $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is a duality pair of complexes.

Proof Let M be an \mathcal{A} complex. Then M is exact and $Z_n(M) \in \mathcal{A}$ for all n . By Lemma 2.2(Yang et al., 2014), we get

$$Z_n(M^+) = \text{Hom}_{\mathbb{Z}}(M_{-n}/B_{-n}(M), \mathbb{Q}/\mathbb{Z}) = (M_{-n}/B_{-n}(M))^+ = (M_{-n}/\text{Ker } \delta_{-n})^+ \cong (\text{Im } \delta_{-n})^+.$$

Since $\text{Im } \delta_{-n} = \text{Ker } \delta_{-n-1} = Z_{-n-1}(M) \in \mathcal{A}$, we get $Z_n(M^+) \in \mathcal{B}$. Note that, M^+ is exact. Therefore, M^+ is a \mathcal{B} complex.

Let M^+ be a \mathcal{B} complex. Then M^+ is exact and $Z_n(M^+) \in \mathcal{B}$ for all n . By Lemma 2.2(Yang et al., 2014), we get

$$Z_n(M^+) = \text{Hom}_{\mathbb{Z}}(M_{-n}/B_{-n}(M), \mathbb{Q}/\mathbb{Z}) = (M_{-n}/B_{-n}(M))^+ \cong (\text{Ker } \delta_{-n-1})^+ = (Z_{-n-1}(M))^+.$$

Note that $(Z_{-n-1}(M))^+ \in \mathcal{B}$, and so $Z_{-n-1}(M) \in \mathcal{A}$. On the other hand, we see that M is exact. Thus, M is a \mathcal{A} complex.

In the following, we will show that $\tilde{\mathcal{B}}$ is closed under direct summands and finite direct sums. Let A, B , and C be complexes, and $A \oplus B = C \in \tilde{\mathcal{B}}$. Obviously, $A \oplus B$ is exact and $Z_n(A \oplus B) \in \mathcal{B}$ for all n . This implies $Z_n(A \oplus B) = Z_n(A) \oplus Z_n(B) \in \mathcal{B}$. Since \mathcal{B} is closed under direct summands, we have $Z_n(A) \in \mathcal{B}$ and $Z_n(B) \in \mathcal{B}$. Thus, A and B are both \mathcal{B} complexes. Similarly, it can be shown that $\tilde{\mathcal{B}}$ is closed under finite direct sums. \square

When investigating a complex G , it is natural to consider the relationship between the complex G and its terms G_n for all $n \in \mathbb{Z}$. Here, we obtain the following result for Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes.

Theorem 1 Let $(\mathcal{A}, \mathcal{B})$ be a duality pair of modules. Then a complex G is Gorenstein $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ -injective if and only if G_n is a Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective module for each $n \in \mathbb{Z}$ and $\text{Hom}_R(B, G)$ is exact for any $B \in \tilde{\mathcal{B}}$, where $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is a duality pair of complexes which is induced by $(\mathcal{A}, \mathcal{B})$.

Proof “ \Rightarrow ” Let G be a Gorenstein $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ -injective complex. Then there exists an exact sequence of injective complexes

$$\cdots \rightarrow I^{-1} \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots$$

with $G = \text{Ker}(I^0 \rightarrow I^1)$ which remains exact after applying $\text{Hom}(B, -)$ for any complex $B \in \tilde{\mathcal{B}}$.

We first show that each G_n is a Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective module. Of course we have the exact complex of injective modules

$$\cdots \rightarrow (I^{-1})_n \rightarrow (I^0)_n \rightarrow (I^1)_n \rightarrow (I^2)_n \rightarrow \cdots$$

with $G_n = \text{Ker}((I^0)_n \rightarrow (I^1)_n)$. So it is left to show that this remains exact after applying $\text{Hom}_R(C, -)$ for all $C \in \mathcal{B}$. Note that $D^n(C)$ is a \mathcal{B} complex for any module $C \in \mathcal{B}$. Using the standard adjunction $\text{Hom}(D^n(C), G) \cong \text{Hom}_R(C, G_n)$ (Gillespie, 2004), we see that the sequence

$$\cdots \rightarrow \text{Hom}_R(C, (I^{-1})_n) \rightarrow \text{Hom}_R(C, (I^0)_n) \rightarrow \text{Hom}_R(C, (I^1)_n) \rightarrow \text{Hom}_R(C, (I^2)_n) \rightarrow \cdots$$

is exact. We conclude that G_n is Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective.

Next we will show that the complex $\text{Hom}_R(B, G)$ is exact for any $B \in \tilde{\mathcal{B}}$. Since G is Gorenstein $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ -

injective it follows from the definition that $\text{Ext}^1(B, G) = 0$. This implies $\text{Ext}_{du}^1(B, G) = 0$. Note that $H_n(\text{Hom}_R(B, G)) \cong \text{Ext}_{du}^1(B, \Sigma^{-n-1}G)$ for any $B \in \tilde{\mathcal{B}}$ (Gillespie, 2004). But it is easy to see $\text{Ext}_{du}^1(B, \Sigma^{-n-1}G) \cong \text{Ext}_{du}^1(\Sigma^{n+1}B, G)$. Since $\Sigma^{n+1}B \in \tilde{\mathcal{B}}$, and so $H_n(\text{Hom}_R(B, G)) \cong \text{Ext}_{du}^1(\Sigma^{n+1}B, G) = 0$ for all $n \in \mathbb{Z}$.

“ \Leftarrow ” Now suppose each G_n is a Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective module and $\text{Hom}_R(B, G)$ is exact for any $B \in \tilde{\mathcal{B}}$. Next, we will prove that G is a Gorenstein $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ -injective complex. We start by using that $\mathcal{C}(R)$ has enough injectives, and write a short exact sequence $0 \rightarrow G \xrightarrow{\delta} I^0 \rightarrow C \rightarrow 0$ where I^0 is an injective complex. Since the class of Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective modules is injectively resolving by Gillespie (2019) Lemma 4. 5, we see that each C_n is also Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective. We claim that C also satisfies that $\text{Hom}_R(B, C)$ is exact for any $B \in \tilde{\mathcal{B}}$. Indeed for any choice of integers n, k and a complex $B \in \tilde{\mathcal{B}}$, we have $\text{Ext}_R^1(B_k, G_{k+n}) = 0$ since $B_k \in \mathcal{B}$ and G_{k+n} is Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective. Then there exists an exact sequence

$$0 \rightarrow \text{Hom}_R(B_k, G_{k+n}) \rightarrow \text{Hom}_R(B_k, (I^0)_{k+n}) \rightarrow \text{Hom}_R(B_k, C_{k+n}) \rightarrow 0.$$

This implies

$$0 \rightarrow \prod_{k \in \mathbb{Z}} \text{Hom}_R(B_k, G_{k+n}) \rightarrow \prod_{k \in \mathbb{Z}} \text{Hom}_R(B_k, (I^0)_{k+n}) \rightarrow \prod_{k \in \mathbb{Z}} \text{Hom}_R(B_k, C_{k+n}) \rightarrow 0$$

is also exact. Hence $0 \rightarrow \text{Hom}_R(B, G) \rightarrow \text{Hom}_R(B, I^0) \rightarrow \text{Hom}_R(B, C) \rightarrow 0$ is an exact sequence of complexes. Since $\text{Hom}_R(B, G)$ and $\text{Hom}_R(B, I^0)$ are exact, it follows that $\text{Hom}_R(B, C)$ is also exact. Since C has the same properties as G , we may inductively obtain an injective coresolution

$$0 \rightarrow G \xrightarrow{\delta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \dots$$

where each $(K^i)_n = (\text{Ker } d^i)_n$ ($i \geq 0$) is a Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective module and which satisfies that $\text{Hom}_R(B, K^i)$ is exact for any complex $B \in \tilde{\mathcal{B}}$. This coresolution must remain exact after applying $\text{Hom}(B, -)$ for any complex $B \in \tilde{\mathcal{B}}$ because $\text{Ext}^1(B, K^i) = \text{Ext}_{du}^1(B, K^i) \cong H_{-1}(\text{Hom}_R(B, K^i)) = 0$ (Gillespie, 2004).

Note that there is an obvious (degreewise split) short exact sequence

$$0 \rightarrow \Sigma^{-1}G \xrightarrow{(1, \delta)} \bigoplus_{n \in \mathbb{Z}} D^n(G_n) \xrightarrow{(\delta, 1)} G \rightarrow 0.$$

Now each G_n is Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective. So we certainly can find a short exact sequence $0 \rightarrow F_n \xrightarrow{\alpha_n} L_n \xrightarrow{\beta_n} G_n \rightarrow 0$ where L_n is injective and F_n is also Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective. This gives us another short exact sequence

$$0 \rightarrow \bigoplus_{n \in \mathbb{Z}} D^n(F_n) \xrightarrow{\bigoplus_{n \in \mathbb{Z}} D^n(\alpha_n)} \bigoplus_{n \in \mathbb{Z}} D^n(L_n) \xrightarrow{\bigoplus_{n \in \mathbb{Z}} D^n(\beta_n)} \bigoplus_{n \in \mathbb{Z}} D^n(G_n) \rightarrow 0.$$

Notice that $\bigoplus_{n \in \mathbb{Z}} D^n(L_n) = \prod_{n \in \mathbb{Z}} D^n(L_n)$ is an injective complex and we will denote it by I^{-1} . Furthermore, let $\iota: I^{-1} \rightarrow G$ be the composite

$$\bigoplus_{n \in \mathbb{Z}} D^n(L_n) \xrightarrow{\bigoplus_{n \in \mathbb{Z}} D^n(\beta_n)} \bigoplus_{n \in \mathbb{Z}} D^n(G_n) \xrightarrow{d+1} G.$$

Then ι is an epimorphism since it is the composite of two epimorphisms. Moreover, setting $K^0 = \text{Ker } \iota$, thus we have the following commutative graph:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{n \in \mathbb{Z}} D^n(F_n) & \longrightarrow & I^{-1} & \longrightarrow & \bigoplus_{n \in \mathbb{Z}} D^n(G_n) \longrightarrow 0 \\ & & \downarrow & & \downarrow \iota & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & G & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} .$$

It follows from the Snake Lemma that K^0 sits in the short exact sequence

$$0 \rightarrow \bigoplus_{n \in \mathbb{Z}} D^n(F_n) \rightarrow K^0 \rightarrow \Sigma^{-1}G \rightarrow 0.$$

Hence $(K^0)_n$ is a Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective module. Because of this, the short exact sequence $0 \rightarrow K^0 \rightarrow I^{-1} \rightarrow G \rightarrow 0$ is $\text{Hom}_R(B, -)$ exact for any $B \in \tilde{\mathcal{B}}$. Then $0 \rightarrow \text{Hom}_R(B, K^0) \rightarrow \text{Hom}_R(B, I^{-1}) \rightarrow \text{Hom}_R(B, G) \rightarrow 0$ is exact. And also $\text{Hom}_R(B, K^0)$ must be exact since $\text{Hom}_R(B, I^{-1})$ and $\text{Hom}_R(B, G)$ are exact. Since K^0 has the same properties as G , we may continue inductively to obtain the desired resolution

$$\dots \rightarrow I^{-3} \xrightarrow{d^{-3}} I^{-2} \xrightarrow{d^{-2}} I^{-1} \xrightarrow{\iota} G \rightarrow 0.$$

Finally, we paste the resolution together with

$$0 \rightarrow G \xrightarrow{\delta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \dots$$

by setting $d^{-1} = \delta\iota$ and we are done. \square

Remark 2 (i) Let \mathcal{L} denote the class of all level modules, and \mathcal{AC} denote the class of all absolutely clean modules, then $(\mathcal{L}, \mathcal{AC})$ is a duality pair of modules, and $(\tilde{\mathcal{L}}, \widetilde{\mathcal{AC}})$ is a duality pair of complexes by Lemma 1.

Thus, Gorenstein $(\tilde{\mathcal{L}}, \widetilde{\mathcal{AC}})$ -injective complexes are exactly Gorenstein \mathcal{AC} -injective.

(ii) Let \mathcal{F} denote the class of all flat modules, and \mathcal{I} denote the class of all injective modules, then $(\mathcal{F}, \mathcal{I})$ is a duality pair of modules, and $(\tilde{\mathcal{F}}, \tilde{\mathcal{I}})$ is a duality pair of complexes by Lemma 1. Thus, Gorenstein $(\tilde{\mathcal{F}}, \tilde{\mathcal{I}})$ -injective complexes are exactly Gorenstein injective.

(iii) Let \mathcal{FPF}_1 denote the class of all FP_1 -flat modules, and \mathcal{FPI}_1 denote the class of all FP_1 -injective modules. If R is a coherent ring, $(\mathcal{FPF}_1, \mathcal{FPI}_1)$ is a duality pair of modules, then $(\widetilde{\mathcal{FPF}}_1, \widetilde{\mathcal{FPI}}_1)$ is a duality pair of complexes by Lemma 1. Thus, Gorenstein $(\widetilde{\mathcal{FPF}}_1, \widetilde{\mathcal{FPI}}_1)$ -injective complexes are exactly Ding injective.

(iv) Let \mathcal{FPF}_n denote the class of all FP_n -flat modules, and \mathcal{FPI}_n denote the class of all FP_n -injective modules. If $n \geq 2$, $(\mathcal{FPF}_n, \mathcal{FPI}_n)$ is a duality pair of modules, then $(\widetilde{\mathcal{FPF}}_n, \widetilde{\mathcal{FPI}}_n)$ is a duality pair of complexes by Lemma 1 and Bravo et al. (2017); Zhao et al. (2019). Thus, Gorenstein $(\widetilde{\mathcal{FPF}}_n, \widetilde{\mathcal{FPI}}_n)$ -injective complexes are exactly Gorenstein FP_n -injective.

Then the following result can be obtained by Theorem 1.

Corollary 1 (Bravo et al., 2016) A complex G is Gorenstein \mathcal{AC} -injective if and only if G_n is a Gorenstein \mathcal{AC} -injective module for each $n \in \mathbb{Z}$ and $\text{Hom}_R(B, G)$ is exact for any absolutely clean complex B .

Corollary 2 Let G be a complex. Then the following conditions are equivalent:

- (i) G is a Gorenstein injective complex.
- (ii) G_n is a Gorenstein injective module for each $n \in \mathbb{Z}$ and $\text{Hom}_R(B, G)$ is exact for any injective complex B .
- (iii) G_n is a Gorenstein injective module for each $n \in \mathbb{Z}$.

Proof (i) \Leftrightarrow (ii) It follows from Theorem 1.

(ii) \Rightarrow (iii) Obviously.

(iii) \Rightarrow (ii) Note that any injective complex is the direct product (sum) of complexes

$$\Sigma^m D(B): \dots \rightarrow 0 \rightarrow B \rightarrow B \rightarrow 0 \rightarrow \dots$$

with B an injective module. Let $0 \rightarrow \Sigma^{-n-1}G \rightarrow X \rightarrow \Sigma^m D(B)$ be an extension in $\mathcal{C}(R)$, and we consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & G_{m+n+3} & \longrightarrow & X_{m+2} & \longrightarrow & 0 \\
 & & \downarrow \delta_{m+n+3}^c & & \downarrow \delta_{m+2}^x & & \downarrow \\
 0 & \longrightarrow & G_{m+n+2} & \xrightarrow{g_{m+n+2}} & X_{m+1} & \xrightarrow{f_{m+1}} & B \longrightarrow 0 \\
 & & \downarrow \delta_{m+n+2}^c & & \downarrow \delta_{m+1}^x & & \downarrow \\
 0 & \longrightarrow & G_{m+n+1} & \xrightarrow{g_{m+n+1}} & X_m & \xrightarrow{f_m} & B \longrightarrow 0 \\
 & & \downarrow \delta_{m+n+1}^c & & \downarrow \delta_m^x & & \downarrow \\
 0 & \longrightarrow & G_{m+n} & \longrightarrow & X_{m-1} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & &
 \end{array}$$

Since $f_{m+1} : X_{m+1} \rightarrow B$ splits, there exists $h_{m+1} : B \rightarrow X_{m+1}$ such that $f_{m+1}h_{m+1} = 1$. We define $h_m : B \rightarrow X_m$ as $h_m = \delta_{m+1}^x h_{m+1}$ and $h_i = 0$ for $i \neq m, m + 1$. In this way, we obtain a map of complexes $h : \Sigma^m D(B) \rightarrow X$ such that $fh = 1$. So the sequence $0 \rightarrow \Sigma^{-n-1}G \rightarrow X \rightarrow \Sigma^m D(B)$ splits, and hence $\text{Ext}^1(\Sigma^m D(B), \Sigma^{-n-1}G) = 0$.

Since

$$\begin{aligned}
 H_n \text{Hom}_R(I, G) &\cong \text{Ext}_{dw}^1(I, \Sigma^{-n-1}G) = \text{Ext}^1(I, \Sigma^{-n-1}G) = \text{Ext}^1\left(\prod_{m \in \mathbb{Z}} \Sigma^m D(B), \Sigma^{-n-1}G\right) \\
 &\cong \prod_{m \in \mathbb{Z}} \text{Ext}^1(\Sigma^m D(B), \Sigma^{-n-1}G) = 0,
 \end{aligned}$$

$\text{Hom}_R(I, G)$ is exact for any injective complex I . □

Corollary 3 A complex G is Gorenstein FP_n -injective if and only if G_n is a Gorenstein FP_n -injective module for each $n \in \mathbb{Z}$ and $\text{Hom}_R(B, G)$ is exact for any FP_n -injective complex B .

Corollary 4 Let G be a complex. Then G is Gorenstein $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ -injective if and only if each G_n is a Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective module for all $n \in \mathbb{Z}$ and any morphism $f : B \rightarrow G$ is null homotopic whenever $B \in \tilde{\mathcal{B}}$.

Proof “ \Rightarrow ” By Theorem 1, we need only to show that $f : B \rightarrow G$ is null homotopic for all $B \in \tilde{\mathcal{B}}$. Since $\text{Hom}_R(B, G)$ is exact for all $B \in \tilde{\mathcal{B}}$, it follows from Lemma 2.1(Yang et al., 2018) that $\text{Hom}_{K(R)}(B, G) \cong H_n(\text{Hom}_R(\Sigma^{-n}B, G)) = 0$. Therefore $f : B \rightarrow G$ is null homotopic whenever $B \in \tilde{\mathcal{B}}$.

“ \Leftarrow ” Since $\Sigma^n B$ is still a \mathcal{B} complex whenever $B \in \tilde{\mathcal{B}}$, it follows from Lemma 2.1(Yang et al., 2018) that $\text{Hom}_R(B, G)$ is exact. □

Proposition 2 Let G be an exact complex bounded below. Suppose that $Z_n(G)$ is a Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective module for each $n \in \mathbb{Z}$. Then G is Gorenstein $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ -injective.

Proof Since the class of Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective modules is injectively resolving, each G_n is a Gorenstein $(\mathcal{A}, \mathcal{B})$ -injective module for all $n \in \mathbb{Z}$. Therefore, by Corollary 4, we need only to show any morphism $f : B \rightarrow G$ is null homotopic whenever $B \in \tilde{\mathcal{B}}$.

We set

$$G := \cdots \rightarrow G_{-n+3} \xrightarrow{\delta_{-n+3}} G_{-n+2} \xrightarrow{\delta_{-n+2}} G_{-n+1} \xrightarrow{\delta_{-n+1}} G_{-n} \xrightarrow{\delta_{-n}} 0 \rightarrow \cdots,$$

where n is a fixed integer. Let $B \in \tilde{\mathcal{B}}$. In the following, we will show that any morphism $f : B \rightarrow G$ is null homotopic.

Consider the following commutative diagram:

$$\begin{array}{ccccccccccc}
 \cdots & \longrightarrow & B_{-n+3} & \xrightarrow{\theta_{-n+3}} & B_{-n+2} & \xrightarrow{\theta_{-n+2}} & B_{-n+1} & \xrightarrow{\theta_{-n+1}} & B_{-n} & \xrightarrow{\theta_{-n}} & B_{-n-1} & \longrightarrow & \cdots \\
 & & \downarrow g_{-n+3} & \swarrow \varepsilon_{-n+2} & \downarrow g_{-n+2} & \swarrow \varepsilon_{-n+1} & \downarrow g_{-n+1} & \swarrow \varepsilon_{-n} & \downarrow g_{-n} & \swarrow 0 & \downarrow g_{-n-1} & & \\
 \cdots & \longrightarrow & G_{-n+3} & \xrightarrow{\delta_{-n+3}} & G_{-n+2} & \xrightarrow{\delta_{-n+2}} & G_{-n+1} & \xrightarrow{\delta_{-n+1}} & G_{-n} & \xrightarrow{\delta_{-n}} & 0 & \longrightarrow & \cdots
 \end{array}$$

with $\varepsilon_k = 0$ for all $k \leq -n - 1$. Then we need to show that there exists a homomorphism $\varepsilon_k : B_k \rightarrow G_{k+1}$ such that $g_k = \varepsilon_{k-1}\theta_k + \delta_{k+1}\varepsilon_k$ for all $k \geq -n$.

Let $Z_{-n+j}(G) = \text{Ker } \delta_{-n+j}, j \geq 1$. Then the sequence

$$0 \longrightarrow Z_{-n+1}(G) \xrightarrow{\mu_{-n+1}} G_{-n+1} \xrightarrow{\delta_{-n+1}} G_{-n} \longrightarrow 0$$

remains exact after applying $\text{Hom}_R(B_{-n}, -)$. This implies

$$0 \rightarrow \text{Hom}_R(B_{-n}, Z_{-n+1}(G)) \rightarrow \text{Hom}_R(B_{-n}, G_{-n+1}) \rightarrow \text{Hom}_R(B_{-n}, G_{-n}) \rightarrow 0$$

is exact. Thus there exists a homomorphism $\varepsilon_{-n} \in \text{Hom}_R(B_{-n}, G_{-n+1})$ such that $g_{-n} = \delta_{-n+1}\varepsilon_{-n}$.

Since

$$\delta_{-n+1}(g_{-n+1} - \varepsilon_{-n}\theta_{-n+1}) = \delta_{-n+1}g_{-n+1} - \delta_{-n+1}\varepsilon_{-n}\theta_{-n+1} = \delta_{-n+1}g_{-n+1} - g_{-n}\theta_{-n+1} = 0,$$

$\text{Im}(g_{-n+1} - \varepsilon_{-n}\theta_{-n+1}) \subseteq \text{Ker } \delta_{-n+1}$. There exists $h_{-n+1} \in \text{Hom}_R(B_{-n+1}, Z_{-n+1}(G))$ such that the following diagram

$$\begin{array}{ccccc}
 & & B_{-n+1} & & \\
 & \swarrow h_{-n+1} & \downarrow g_{-n+1} - \varepsilon_{-n}\theta_{-n+1} & & \\
 Z_{-n+1}(G) & \xrightarrow{\mu_{-n+1}} & G_{-n+1} & \xrightarrow{\delta_{-n+1}} & G_{-n}
 \end{array}$$

commutes. That is, we have the equality $g_{-n+1} - \varepsilon_{-n}\theta_{-n+1} = \mu_{-n+1}h_{-n+1}$.

On the other hand, we see that the sequence

$$0 \longrightarrow Z_{-n+2}(G) \xrightarrow{\mu_{-n+2}} G_{-n+2} \xrightarrow{\pi_{-n+2}} Z_{-n+1}(G) \longrightarrow 0$$

remains exact after applying $\text{Hom}_R(B_{-n+1}, -)$. This implies

$$0 \rightarrow \text{Hom}_R(B_{-n+1}, Z_{-n+2}(G)) \rightarrow \text{Hom}_R(B_{-n+1}, G_{-n+2}) \rightarrow \text{Hom}_R(B_{-n+1}, Z_{-n+1}(G)) \rightarrow 0$$

is exact. Thus there also exists a homomorphism $\varepsilon_{-n+1} \in \text{Hom}_R(B_{-n+1}, G_{-n+2})$ such that the following diagram

$$\begin{array}{ccccc}
 & & B_{-n+1} & & \\
 & \swarrow \varepsilon_{-n+1} & \downarrow h_{-n+1} & & \\
 G_{-n+2} & \xrightarrow{\pi_{-n+2}} & Z_{-n+1}(G) & \longrightarrow & 0
 \end{array}$$

commutes. That is, we have the equality $h_{-n+1} = \pi_{-n+2}\varepsilon_{-n+1}$.

Thus $g_{-n+1} - \varepsilon_{-n}\theta_{-n+1} = \mu_{-n+1}\pi_{-n+2}\varepsilon_{-n+1}$. Note that $\mu_{-n+1}\pi_{-n+2} = \delta_{-n+2}$, and so $g_{-n+1} = \varepsilon_{-n}\theta_{-n+1} + \delta_{-n+2}\varepsilon_{-n+1}$.

Hence, we may continue inductively to obtain that $g_k = \varepsilon_{k-1}\theta_k + \delta_{k+1}\varepsilon_k$, where $\varepsilon_k : B_k \rightarrow G_{k+1}$ for any $k \geq -n$. Therefore $f : B \rightarrow G$ is null homotopic. □

Lemma 2 Let \mathcal{X} be a class of complexes which is either projectively resolving, or injectively resolving. If \mathcal{X} is closed under countable direct sums, or closed under countable direct products, then \mathcal{X} is also closed under direct summands.

Proof Assume that Y is a direct summand of $X \in \mathcal{X}$. We wish to show that $Y \in \mathcal{X}$. Write $X = Y \oplus Z$ for some complex Z . If \mathcal{X} is closed under countable direct sums, then we define $W = Y \oplus Z \oplus Y \oplus Z \oplus \cdots$ (direct sum), and note that $W \cong X \oplus X \oplus \cdots \in \mathcal{X}$. If \mathcal{X} is closed under countable direct products, then we put $W = Y \times Z \times Y \times Z \times \cdots$ (direct product), and note that $W \cong X \times X \times \cdots \in \mathcal{X}$. In either case we have $W \cong Y \oplus W$, so in particular the sum $Y \oplus W$ belongs to \mathcal{X} . If \mathcal{X} is projectively resolving, then we consider the split exact sequence $0 \rightarrow Y \rightarrow Y \oplus W \rightarrow W \rightarrow 0$, and if \mathcal{X} is injectively resolving, then we consider $0 \rightarrow W \rightarrow W \oplus Y \rightarrow Y \rightarrow 0$. In either case we conclude that $Y \in \mathcal{X}$. □

Proposition 3 Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of complexes. Then the following hold:

- (i) The class of all Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes is injectively resolving.

(ii) The class of all Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes is closed under products and direct summands, respectively.

(iii) If $M, M'' \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$ and $\text{Ext}^1(I, M') = 0$ for any injective complex I , then $M' \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$.

Proof (i) Let M' and M'' be Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes. Then

$$\text{Ext}^{\geq 1}(Y, M') = \text{Ext}^{\geq 1}(Y, M'') = 0 \text{ for } Y \in \mathcal{Y}.$$

Note that there exists a long exact sequence

$$\dots \rightarrow \text{Ext}^i(Y, M') \rightarrow \text{Ext}^i(Y, M) \rightarrow \text{Ext}^i(Y, M'') \rightarrow \dots,$$

we get $\text{Ext}^{\geq 1}(Y, M) = 0$. By Proposition 1(ii) and Lemma 8.2.1 (Enochs et al., 2000), we get $M \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$.

Suppose $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of complexes with M' and M Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes. We need only to show that M'' is a Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex. Since $M \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$, there exists a short exact sequence $0 \rightarrow K \rightarrow I \rightarrow M \rightarrow 0$, where I is injective and K is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective. Then we have the pullback diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & K & \xlongequal{\quad} & K & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & I & \longrightarrow & M'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Note that K and M' are both Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective. Then U is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective. Thus M'' is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective by Proposition 1.

(ii) Since the class of injective complexes is closed under direct products, we have Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex is closed under product. Note that Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex is closed under direct summands by (1) and Lemma 2.

(iii) Since $M'' \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$, there exists an exact sequence of complexes $0 \rightarrow K \rightarrow I \rightarrow M'' \rightarrow 0$ with $I \in \mathcal{I}, K \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. Consider the following pullback diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & K & \xlongequal{\quad} & K & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & U & \longrightarrow & I \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Note that M and K are Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective, and so U is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective by (i). Since $\text{Ext}^1(I, M') = 0$, the middle row split. Therefore, M' is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective by (ii). \square

Proposition 4 Consider a short exact sequence of complexes $0 \rightarrow M \xrightarrow{f} A \rightarrow N \rightarrow 0$, where $A \in \mathcal{I}, N \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. Then, for any homomorphism $f': M \rightarrow A'$, $\text{Coker } \alpha \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$, where $A' \in \mathcal{I}, \alpha = (f, f')$: $M \rightarrow A \oplus A'$.

Proof Let $f': M \rightarrow A'$ with $A' \in \mathcal{I}$. Consider the short exact sequence of complexes

$0 \rightarrow M \xrightarrow{\alpha} A \oplus A' \rightarrow \text{Coker } \alpha \rightarrow 0$. By diagram-chasing, there exists a homomorphism $\mu: \text{Coker } \alpha \rightarrow N$ that makes the following diagram commute:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\alpha} & A \oplus A' & \longrightarrow & \text{Coker } \alpha \longrightarrow 0 \\ & & \parallel & & \downarrow \pi & & \downarrow \mu \\ 0 & \longrightarrow & M & \longrightarrow & A & \longrightarrow & N \longrightarrow 0 \end{array}$$

where $\pi: A \oplus A' \rightarrow A$ is a classical projection. Applying the Snake Lemma yields that μ is an epimorphism and $\text{Ker } \mu = \text{Ker } \pi = A' \in \mathcal{I}$. Furthermore, by Proposition 3, it follows that $\text{Coker } \alpha \in \mathcal{GI}_{(x,y)}(R)$ in the exact sequence $0 \rightarrow \text{Ker } \mu \rightarrow \text{Coker } \alpha \rightarrow N \rightarrow 0$. \square

Lemma 3 A complex M has a $\text{Hom}(\mathcal{Y}, -)$ exact injective resolution if and only if it has a $\text{Hom}(\mathcal{Y}, -)$ exact $\mathcal{GI}_{(x,y)}(R)$ resolution.

Proof “ \Rightarrow ” Clearly.

“ \Leftarrow ” Let $0 \rightarrow K \rightarrow G^0 \rightarrow M \rightarrow 0$ be a $\text{Hom}(\mathcal{Y}, -)$ exact exact sequence of complexes with $G^0 \in \mathcal{GI}_{(x,y)}(R)$ and K having a $\text{Hom}(\mathcal{Y}, -)$ exact $\mathcal{GI}_{(x,y)}(R)$ resolution. Then we have the following pullback diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & G' & \xlongequal{\quad} & G' & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H & \longrightarrow & I^1 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & K & \longrightarrow & G^0 & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with $I^1 \in \mathcal{I}$, $G' \in \mathcal{GI}_{(x,y)}(R)$. Since the bottom row and the middle column are $\text{Hom}(\mathcal{Y}, -)$ exact, so is the middle row by Snake Lemma. Note that there is an $\text{Hom}(\mathcal{Y}, -)$ exact exact sequence of complexes $0 \rightarrow K^1 \rightarrow G^1 \rightarrow K \rightarrow 0$ such that K^1 has a $\text{Hom}(\mathcal{Y}, -)$ exact $\mathcal{GI}_{(x,y)}(R)$ resolution and $G^1 \in \mathcal{GI}_{(x,y)}(R)$. Consider the following pullback diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & K & \xlongequal{\quad} & K & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & U & \longrightarrow & I \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Since $\mathcal{GI}_{(x,y)}(R)$ is closed under extensions, $L \in \mathcal{GI}_{(x,y)}(R)$. Therefore, H has a $\text{Hom}(\mathcal{Y}, -)$ exact $\mathcal{GI}_{(x,y)}(R)$ resolution. Note that $0 \rightarrow H \rightarrow I^1 \rightarrow M \rightarrow 0$ is $\text{Hom}(\mathcal{Y}, -)$ exact. Continuing the process above, we can get that M has a $\text{Hom}(\mathcal{Y}, -)$ exact injective resolution. \square

Dually, we can prove the following lemma.

Lemma 4 A complex M has a $\text{Hom}(\mathcal{Y}, -)$ exact injective coresolution if and only if it has a $\text{Hom}(\mathcal{Y}, -)$ ex-

act $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$ coresolution.

Definition 6 Let $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}^2(R)$ denote the class of complexes M for which there exists an exact sequence of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes

$$\dots \rightarrow G^{-1} \rightarrow G^0 \rightarrow G^1 \rightarrow G^2 \rightarrow \dots$$

such that $M \cong \text{Ker}(G^0 \rightarrow G^1)$, and such that $\text{Hom}(Y, -)$ leaves the sequence exact for all $Y \in \mathcal{Y}$.

It is obvious that $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R) \subseteq \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}^2(R)$. As a consequence of Lemma 3 and Lemma 4, we have the following result.

Theorem 2 $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R) = \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}^2(R)$.

3 Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective dimension

In this section we give a detailed treatment for Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective dimension of complexes.

Definition 7 Let \mathcal{D} be a class of complexes. The \mathcal{D} injective dimension of a complex M is defined to be the infimum of integers n such that there exists an exact sequence

$$0 \rightarrow M \rightarrow D^0 \rightarrow D^1 \rightarrow \dots \rightarrow D^n \rightarrow 0$$

with all $D^i \in \mathcal{D}$, and we write $\mathcal{D}\text{-id}(M) \leq n$. If no such finite sequence exists, define $\mathcal{D}\text{-id}(M) = +\infty$.

In particular, we use $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M)$ to denote the Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective dimension of M when $\mathcal{D} = \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$.

Proposition 5 Let M be a Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex. Then for any complex L with finite \mathcal{Y} injective dimension,

$$\text{Ext}^{\geq 1}(L, M) = 0.$$

Proof Assume that L is a complex and $\mathcal{Y}\text{-id}(L) = n, 1 \leq n < +\infty$. Then there exists an exact sequence of complexes

$$0 \rightarrow L \rightarrow Y^0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} \dots \rightarrow Y^{n-1} \xrightarrow{d^{n-1}} Y^n \rightarrow 0,$$

where each $Y^i \in \mathcal{Y}$. By dimension shifting, we get

$$\text{Ext}^i(L, M) \cong \text{Ext}^{i+1}(\text{Ker } d^1, M) \cong \dots \cong \text{Ext}^{i+n}(Y^n, M), \quad i \geq 1.$$

Note that $\text{Ext}^{i+n}(Y^n, M) = 0$, and so $\text{Ext}^i(L, M) = 0, i \geq 1$. □

Corollary 5 Consider an exact sequence $0 \rightarrow M \rightarrow G^0 \rightarrow \dots \rightarrow G^{n-1} \rightarrow K^n \rightarrow 0$ where G^0, \dots, G^{n-1} are Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes. Then

$$\text{Ext}^{i+n}(L, M) \cong \text{Ext}^i(L, K^n)$$

for all complexes L with finite \mathcal{Y} injective dimension, and all $i \in \mathbb{N}$.

Proof It is easily seen by Proposition 5 and dimension shifting. □

Theorem 3 Assume that M is a complex, and $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M) = n < +\infty$. Then there are exact sequences

$$0 \rightarrow M \rightarrow G \rightarrow I \rightarrow 0, \quad 0 \rightarrow G' \rightarrow I' \rightarrow M \rightarrow 0,$$

such that $G, G' \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$, $\text{id}(I') \leq n, \text{id}(I) \leq n - 1$.

Proof We proceed by induction on $n \geq 0$. If $n = 0$, then M is a Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex. By Proposition 1(iii), we have the following two exact sequences of complexes:

$$0 \rightarrow M \rightarrow M \rightarrow 0 \rightarrow 0, \quad 0 \rightarrow G' \rightarrow I' \rightarrow M \rightarrow 0,$$

where $I' \in \mathcal{I}, G' \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. If $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M) = n > 0$, then there exists an exact sequence

$$0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \dots \rightarrow G^n \rightarrow 0,$$

such that each $G^i \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. Assume that $K^1 = \text{Im}(G^0 \rightarrow G^1)$. Then there exist exact sequences of the form:

$$0 \rightarrow M \rightarrow G^0 \rightarrow K^1 \rightarrow 0, \quad 0 \rightarrow K^1 \rightarrow G^1 \rightarrow \dots \rightarrow G^n \rightarrow 0.$$

Therefore, $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(K^1) \leq n - 1$. By the induction hypothesis, there exists an exact sequence

$0 \rightarrow G'' \rightarrow I \rightarrow K^1 \rightarrow 0$ such that $\text{id}(I) \leq n - 1$, $G'' \in \mathcal{GI}_{(x,y)}(R)$. Consider the following pullback diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & G'' & \xlongequal{\quad} & G'' & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & I \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & G^0 & \longrightarrow & K^1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

$G \in \mathcal{GI}_{(x,y)}(R)$ by Proposition 3(i). Obviously, we also have the following exact sequence

$$0 \rightarrow G' \rightarrow I'' \rightarrow G \rightarrow 0,$$

where $I'' \in \mathcal{I}$, $G' \in \mathcal{GI}_{(x,y)}(R)$. Consider the following pullback diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & G' & \xlongequal{\quad} & G' & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & I' & \longrightarrow & I'' & \longrightarrow & I \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & I \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

From the exactness in the middle row of the above diagram, it follows that $\text{id}(I') \leq n$. Therefore, $0 \rightarrow M \rightarrow G \rightarrow I \rightarrow 0$ and $0 \rightarrow G' \rightarrow I' \rightarrow M \rightarrow 0$ are obtained. □

Proposition 6 Let $0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow K \rightarrow 0$ be an exact sequence of complexes. If $G^0, G^1 \in \mathcal{GI}_{(x,y)}(R)$, then there are exact sequences

$$0 \rightarrow M \rightarrow G \rightarrow I \rightarrow K \rightarrow 0, \quad 0 \rightarrow M \rightarrow I' \rightarrow G' \rightarrow K \rightarrow 0,$$

such that $I, I' \in \mathcal{I}$, $G, G' \in \mathcal{GI}_{(x,y)}(R)$.

Proof Since $G^1 \in \mathcal{GI}_{(x,y)}(R)$, there exists an exact sequence of complexes $0 \rightarrow G^2 \rightarrow I \rightarrow G^1 \rightarrow 0$, with $I \in \mathcal{I}$, $G^2 \in \mathcal{GI}_{(x,y)}(R)$. Assume that $L = \text{Im}(G^0 \rightarrow G^1)$. Consider the following pullback diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & G^2 & \xlongequal{\quad} & G^2 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & U & \longrightarrow & I & \longrightarrow & K \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & L & \longrightarrow & G^1 & \longrightarrow & K \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

and pullback diagram:

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
& & & G^2 & \xlongequal{\quad} & G^2 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & U \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M & \longrightarrow & G^0 & \longrightarrow & L \longrightarrow 0 \\
& & & & \downarrow & & \downarrow & \\
& & & & 0 & & 0 & .
\end{array}$$

Since $G^0, G^2 \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$, it follows from Proposition 3(i) that $G \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. Therefore, the desired exact sequence $0 \rightarrow M \rightarrow G \rightarrow I \rightarrow K \rightarrow 0$ is obtained from the short exact sequence $0 \rightarrow M \rightarrow G \rightarrow U \rightarrow 0$ and $0 \rightarrow U \rightarrow I \rightarrow K \rightarrow 0$, where $I \in \mathcal{I}, G \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$.

Similarly, there exists an exact sequence $0 \rightarrow M \rightarrow I' \rightarrow G' \rightarrow K \rightarrow 0$ such that $I' \in \mathcal{I}, G' \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. \square

Theorem 4 Let M be a complex with finite Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective dimension and n a non-negative integer. Then the following conditions are equivalent:

- (i) $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M) \leq n$.
- (ii) $\text{Ext}^{>n}(L, M) = 0$, and all complexes L with finite \mathcal{Y} injective dimension.
- (iii) $\text{Ext}^{>n}(Y, M) = 0$, and all complexes $Y \in \mathcal{Y}$.
- (iv) For every exact sequence of complexes

$$0 \rightarrow M \rightarrow G^0 \rightarrow \dots \rightarrow G^{n-1} \rightarrow K^n \rightarrow 0,$$

where $G^i \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R), \mathcal{I} \in \mathcal{Y}$, then also $K^n \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$.

- (v) For every non-negative integer t such that $0 \leq t \leq n$, there exists an exact sequence of complexes

$$0 \rightarrow M \rightarrow C^0 \rightarrow \dots \rightarrow C^t \rightarrow \dots \rightarrow C^n \rightarrow 0,$$

such that $C^i \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R), C^i \in \mathcal{I} (i \neq t)$.

- (vi) For every non-negative integer t such that $0 \leq t \leq n$, there exists an exact sequence of complexes

$$0 \rightarrow M \rightarrow C^0 \rightarrow \dots \rightarrow C^t \rightarrow \dots \rightarrow C^n \rightarrow 0,$$

such that $C^i \in \mathcal{I}, C^i \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R) (i \neq t)$.

- (vii) For every non-negative integer t such that $1 \leq t \leq n$, there exists an exact sequence of complexes

$$0 \rightarrow M \rightarrow C^0 \rightarrow \dots \rightarrow C^t \rightarrow \dots \rightarrow C^n \rightarrow 0,$$

such that $C^i \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R) (0 \leq i < t), C^j \in \mathcal{I} (t \leq j \leq n)$.

- (viii) For every non-negative integer t such that $1 \leq t \leq n$, there exists an exact sequence of complexes

$$0 \rightarrow M \rightarrow C^0 \rightarrow \dots \rightarrow C^t \rightarrow \dots \rightarrow C^n \rightarrow 0,$$

such that $C^i \in \mathcal{I} (0 \leq i < t), C^j \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R) (t \leq j \leq n)$.

Proof (i) \Rightarrow (ii) We assume that $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M) \leq n$. By definition there is an exact sequence $0 \rightarrow M \rightarrow G^0 \rightarrow \dots \rightarrow G^{n-1} \rightarrow G^n \rightarrow 0$ where G^0, \dots, G^n are Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes. By Proposition 5, we get $\text{Ext}^1(L, G^i) = 0$ for all complexes L with finite \mathcal{Y} injective dimension. By Corollary 5, we have $\text{Ext}^{j+n}(L, M) \cong \text{Ext}^j(L, G^n) = 0, j \in \mathbb{N}$, and consequently $\text{Ext}^{>n}(L, M) = 0$.

(ii) \Rightarrow (iii) Obviously.

(iii) \Rightarrow (iv) We consider an exact sequence

$$0 \rightarrow M \rightarrow G^0 \rightarrow \dots \rightarrow G^{n-1} \rightarrow K^n \rightarrow 0,$$

where G^0, \dots, G^{n-1} are Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes. Applying Corollary 5 to this sequence, and using the assumption, we get that $\text{Ext}^j(Y, K^n) \cong \text{Ext}^{j+n}(Y, M) = 0$ for every $Y \in \mathcal{Y}$, and every $j \in \mathbb{N}$. Since $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$ -

$\text{id}(M) < +\infty$, $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(K^n) < +\infty$. Then there exists an exact sequence

$$0 \rightarrow K^n \rightarrow H^0 \rightarrow \dots \rightarrow H^{m-1} \rightarrow H^m \rightarrow 0,$$

where H^0, \dots, H^m are Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes. Assume that $C^j = \text{Ker}(H^j \rightarrow H^{j+1})$, $j = 0, 1, \dots, m-1$. By Corollary 5, we get $\text{Ext}^1(Y, C^j) \cong \text{Ext}^{j+1}(Y, K^n) = 0$. By Proposition 3(iii), we have $C^j \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. In particular, $K^n = C^0 \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$.

(iv) \Rightarrow (i) and (v) \Rightarrow (i) is obviously.

(i) \Rightarrow (v) We proceed by induction on n . Suppose $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M) \leq 1$. Then there exists an exact sequence $0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow 0$ in $\mathcal{C}(R)$ with G^0 and G^1 Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective. By Proposition 6 with $K = 0$, we get the exact sequences $0 \rightarrow M \rightarrow G \rightarrow I \rightarrow 0$ and $0 \rightarrow M \rightarrow I' \rightarrow G' \rightarrow 0$ such that $I, I' \in \mathcal{I}, G, G' \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$.

Now suppose $n \geq 2$. There exists an exact sequence of complexes $0 \rightarrow M \rightarrow X^0 \rightarrow X^1 \rightarrow \dots \rightarrow X^n \rightarrow 0$ with $X^i \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$ for $1 \leq i \leq n$. Set $K^2 = \text{Ker}(X^2 \rightarrow X^3)$. By applying Proposition 6 to the exact sequence $0 \rightarrow M \rightarrow X^0 \rightarrow X^1 \rightarrow K^2 \rightarrow 0$, we get an exact sequence of complexes $0 \rightarrow M \rightarrow I^0 \rightarrow \widetilde{X}^1 \rightarrow X^2 \rightarrow \dots \rightarrow X^n \rightarrow 0$ with I^0 injective and \widetilde{X}^1 Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective. Set $K' = \text{Ker}(\widetilde{X}^1 \rightarrow X^2)$. Then we have $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(K') \leq n-1$. By the induction hypothesis, there exists an exact sequence of complexes $0 \rightarrow M \rightarrow I^0 \rightarrow G^1 \rightarrow \dots \rightarrow G^t \rightarrow \dots \rightarrow G^n \rightarrow 0$ such that I^0 is injective and G^t is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective and G^i is injective for $i \neq t$ and $1 \leq t \leq n$.

Now we need only to prove (v) for $t = 0$. Set $K^1 = \text{Ker}(X^1 \rightarrow X^2)$. By the induction hypothesis, we get an exact sequence $0 \rightarrow K^1 \rightarrow \widetilde{X}^1 \rightarrow I^2 \rightarrow I^3 \rightarrow \dots \rightarrow I^n \rightarrow 0$ with \widetilde{X}^1 Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective and I^i injective for any $2 \leq i \leq n$. Set $K'' = \text{Ker}(I^2 \rightarrow I^3)$. Then by applying Proposition 6 to the exact sequence $0 \rightarrow M \rightarrow X^0 \rightarrow \widetilde{X}^1 \rightarrow K'' \rightarrow 0$, we get an exact sequence of complexes $0 \rightarrow M \rightarrow \widetilde{X}^0 \rightarrow I^1 \rightarrow K'' \rightarrow 0$ with \widetilde{X}^0 Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective and I^1 injective. Thus we obtain the desired exact sequence $0 \rightarrow \widetilde{X}^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots \rightarrow I^n \rightarrow 0$.

The proof method similar to (i) \Leftrightarrow (v) yields (i) \Leftrightarrow (vi), (i) \Leftrightarrow (vii), (i) \Leftrightarrow (viii). □

Corollary 6 Let M be a complex and $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M) < +\infty$. Then

$$\begin{aligned} \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M) &= \sup\{i \in \mathbb{N} \mid \text{there is a complex } L \text{ and } \mathcal{Y}\text{-id}(L) < +\infty \text{ such that } \text{Ext}^i(L, M) \neq 0\} \\ &= \sup\{i \in \mathbb{N} \mid \text{for } Y \in \mathcal{Y} \text{ such that } \text{Ext}^i(Y, M) \neq 0\}. \end{aligned}$$

Proof According to the equivalence of Theorem 4(i)-(iii), it can be proved. □

Proposition 7 If $(M^i)_{i \in I}$ is a family of complexes, then we have an equality

$$\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(\prod M^i) = \sup\{\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M^i) \mid i \in I\}.$$

Proof The inequality “ \leq ” is clear since $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$ is closed under product.

For the converse inequality “ \geq ”, it suffices to show that if M' is any direct summand of a complex M , then $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M') \leq \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M)$. Assume that $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M) = n < +\infty$, and then proceed by induction on n .

If $n = 0$, $M \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. By Proposition 3(ii), $M' \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$, so $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M') = \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M) = 0$. If $n > 0$, we write $M = M' \oplus M''$ for some complex M'' . Pick exact sequence of complexes $0 \rightarrow M' \rightarrow I' \rightarrow K' \rightarrow 0$ and $0 \rightarrow M'' \rightarrow I'' \rightarrow K'' \rightarrow 0$, where I' and I'' are injective complexes. We get a commutative diagram with split-exact rows,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I' & \longrightarrow & I' \oplus I'' & \longrightarrow & I'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K' & \longrightarrow & K' \oplus K'' & \longrightarrow & K'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Because the middle column is an exact sequence, so we get that $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(K' \oplus K'') = n - 1$. Hence the induction hypothesis yields that $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(K') \leq \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(K' \oplus K'') = n - 1$, and thus the short exact sequence $0 \rightarrow M' \rightarrow I' \rightarrow K' \rightarrow 0$ shows that $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)\text{-id}(M') \leq n$, as desired. \square

Lemma 5 Let M be a complex,

$$0 \rightarrow M \rightarrow G^0 \rightarrow \dots \rightarrow G^{n-1} \rightarrow K^n \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow I^0 \rightarrow \dots \rightarrow I^{n-1} \rightarrow \hat{K}^n \rightarrow 0$$

two exact sequences with G^i Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes and I^i injective complexes for $0 \leq i \leq n - 1$. Then K^n is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective if and only if \hat{K}^n is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective.

Proof There exist homomorphisms, $G^i \rightarrow I^i$ for $0 \leq i \leq n - 1$, and $K^n \rightarrow \hat{K}^n$, such that the following diagram is commutative:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & M & \longrightarrow & G^0 & \longrightarrow & G^1 & \longrightarrow & \dots & \longrightarrow & G^{n-1} & \longrightarrow & K^n & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots & \longrightarrow & I^{n-1} & \longrightarrow & \hat{K}^n & \longrightarrow & 0
 \end{array}$$

This diagram gives a chain map between complexes:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & G^0 & \longrightarrow & G^1 & \longrightarrow & \dots & \longrightarrow & G^{n-1} & \longrightarrow & K^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots & \longrightarrow & I^{n-1} & \longrightarrow & \hat{K}^n & \longrightarrow & 0
 \end{array}$$

Thus, by constructing mapping cones, we obtain an exact sequence of complexes

$$0 \rightarrow G^0 \rightarrow I^0 \oplus G^1 \rightarrow \dots \rightarrow I^{n-2} \oplus G^{n-1} \rightarrow I^{n-1} \oplus K^n \rightarrow \hat{K}^n \rightarrow 0,$$

where $G^0 \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$, and $I^i \oplus G^{i+1} \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$ for $i = 0, 1, \dots, n - 2$.

By Proposition 3, the class of all Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes is injectively resolving, and hence, $K^{n-1} = \text{Ker}(I^{n-1} \oplus K^n \rightarrow \hat{K}^n) \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. Then there exists an exact sequence of complexes

$$0 \rightarrow K^{n-1} \rightarrow I^{n-1} \oplus K^n \rightarrow \hat{K}^n \rightarrow 0,$$

where $K^n \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$ if and only if $\hat{K}^n \in \mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}(R)$. \square

Corollary 7 Let M be a complex,

$$0 \rightarrow M \rightarrow G^0 \rightarrow \dots \rightarrow G^{n-1} \rightarrow K^n \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow \hat{G}^0 \rightarrow \dots \rightarrow \hat{G}^{n-1} \rightarrow \hat{K}^n \rightarrow 0$$

two exact sequence of complexes. If G^i and \hat{G}^i are Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes for $0 \leq i \leq n - 1$, then K^n is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective if and only if \hat{K}^n is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective.

Proof This follows from Lemma 5. □

Below, we will consider a class of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes, where $\mathcal{I} \in \mathcal{Y}$.

Theorem 5 Let M be a complex with finite Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective dimension n . Then M admits an injective Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective preenvelope, $\phi : M \rightarrow G$, where $C = \text{Coker } \phi$ satisfies $\text{id}(\text{Coker } \phi) = n - 1$.

Proof Pick an exact sequence, $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots \rightarrow I^{n-1} \rightarrow C \rightarrow 0$, where I^i are injectives. Then C is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective by Lemma 9. According to the definition of Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complexes, there exists an exact sequence of complexes $0 \rightarrow K \rightarrow E^{-n} \rightarrow E^{-n+1} \rightarrow \dots \rightarrow E^{-1} \rightarrow C \rightarrow 0$, where E^{-n}, \dots, E^{-1} are injectives, K is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective, and such that the functor $\text{Hom}(Y, -)$ leaves this sequence exact for all $Y \in \mathcal{Y}$.

Thus there exist homomorphisms, $I^i \rightarrow E^{i-n}$ for $i = 0, \dots, n - 1$, and $M \rightarrow K$, such that the following diagram is commutative:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & M & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots & \longrightarrow & I^{n-1} & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \parallel & & \\
 0 & \longrightarrow & K & \longrightarrow & E^{-n} & \longrightarrow & E^{-n+1} & \longrightarrow & \dots & \longrightarrow & E^{-1} & \longrightarrow & C & \longrightarrow & 0.
 \end{array}$$

This diagram gives a chain map between complexes:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & M & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots & \longrightarrow & I^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 0 & \longrightarrow & K & \longrightarrow & E^{-n} & \longrightarrow & E^{-n+1} & \longrightarrow & \dots & \longrightarrow & E^{-1} & \longrightarrow & 0,
 \end{array}$$

which induces an isomorphism in homology. Its mapping cone is exact, and all the complexes in it, except for $I^0 \oplus K$ (which is Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective), are injective. Hence the cokernel C of $\phi : M \rightarrow I^0 \oplus K$ satisfies $\text{id}(\text{Coker } \phi) = n - 1$.

Since C has finite injective dimension, we have $\text{Ext}^{\geq 1}(C, M') = 0$ for any Gorenstein $(\mathcal{X}, \mathcal{Y})$ -injective complex M' , by Proposition 5. Hence $\phi : M \rightarrow G$ is the desired preenvelope of M . □

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关于对偶对的 Gorenstein 同调复形

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摘要: 设 $(\mathcal{X}, \mathcal{Y})$ 是复形的对偶对. 引入并研究了 Gorenstein $(\mathcal{X}, \mathcal{Y})$ -内射复形. 对一些特殊的对偶对 $(\mathcal{X}, \mathcal{Y})$, 建立了 Gorenstein $(\mathcal{X}, \mathcal{Y})$ -内射复形和它的层次模之间的关系. 最后, 得到了复形的 Gorenstein $(\mathcal{X}, \mathcal{Y})$ -内射维数的一些等价刻画.

关键词: Gorenstein $(\mathcal{X}, \mathcal{Y})$ -内射复形; 同调性质; Gorenstein $(\mathcal{X}, \mathcal{Y})$ -内射维数

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